Relation between Real and Complex Groups with Respect to their Structure and Continuity.

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Let G_r denote a given r-parameter group, generated by the r infinitesimal transformations $X_1, \ldots X_r$, where

$$X_j \equiv \sum_1^r {}_k \xi_{jk}(x_1 \ldots x_r) \frac{\partial}{\partial x_k}, \qquad (j = 1, 2, \ldots r).$$

If the finite equations defining a transformation T_a of this group are not the canonical equations of the group, the form of the infinitesimal transformation by which T_a is generated is not apparent.* It may, however, be obtained as follows: In order to transform the finite equations of T_a into their canonical form, it is necessary to introduce new parameters $\mu_1 \ldots \mu_r$ (the so-called canonical parameters) defined by equations of the form

$$\mu_k = N_k(a_1, a_2, \ldots a_r), \qquad (k = 1, 2, \ldots r),$$

which are obtained from the finite equations defining T_a by a process of differentiation, elimination and integration.† Then the transformation T_μ of the given group, in the canonical parameters $\mu_1 \ldots \mu_r$, is generated by the infinitesimal transformation

$$U_\mu \equiv \mu_1 X_1 + \ldots + \mu_r X_r.$$

Consequently, if we replace the μ's in this infinitesimal transformation by their functional values in terms of the a's, we obtain the infinitesimal transformation

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* Man kann aber nicht so leicht einsehen welche infinitesimale Transformationen gerade eine der gefundenen endliche Transformationen erzeugt. Lie, Continuierliche Gruppen, p. 195.

† Lie, Transformations Gruppen, Vol. 3, pp. 609–11.

by which T_a is generated, namely,

$$U_a \equiv N_1(a)X_1 + \ldots\ldots + N_r(a)X_r.$$

For certain systems of values of the a's, say $\bar{a}_1 \ldots\ldots \bar{a}_r$, one or more of the functions $N_k(\bar{a})$ $(k = 1, 2, \ldots\ldots r)$ may be infinite in all branches. If such is the case, the transformation $U_{\bar{a}}$ is no longer infinitesimal, and, consequently, the transformation $T_{\bar{a}}$ is not generated by an infinitesimal transformation of the group. This is the briefest possible explanation of the well-known fact that discontinuity may occur in a group with continuous parameters.*

To illustrate what precedes, consider the finite equations

$$x_1' = x_1 e^{a_3} + a_1 e^{a_3}, \quad x_2' = x_2 + a_2, \quad x_3' = x_3 + a_3 \tag{1}$$

which define a transformation T_a of the group G_3 generated by the infinitesimal transformations whose symbols are $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, x_1\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}$. Carrying out on T_a the process given by Lie for transforming these equations into their canonical form, we finally obtain the system of equations

$$\mu_1 = \frac{a_1 a_3 e^{a_3}}{e^{a_3} - 1} \equiv N_1(a), \quad \mu_2 = a_2 \equiv N_2(a), \quad \mu_3 = a_3 \equiv N_3(a),$$

which define the canonical parameters μ_1, μ_2, μ_3 in terms of a_1, a_2, a_3. A transformation T_μ of the group, in its canonical form, is generated by the infinitesimal transformation

$$U_\mu \equiv \mu_1 \frac{\partial}{\partial x_1} + \mu_2 \frac{\partial}{\partial x_2} + \mu_3\left(x_1\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}\right).$$

Consequently, the transformation T_a, defined by equations (1); is generated by the infinitesimal transformation

$$U_a \equiv \frac{a_1 a_3 e^{a_3}}{e^{a_3} - 1}\frac{\partial}{\partial x_1} + a_2\frac{\partial}{\partial x_2} + a\left(x_1\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}\right).$$

To verify this result, the finite equations generated by U_a can be obtained by summation of the infinite series

* Proc. Amer. Acad., Vol. 35, pp. 239–250, 483–485.

$$x_i' = x_i + \sum_1^r {}_j N^j(a) X_j x_i + \frac{1}{2!} \sum_1^r {}_j \sum_1^r {}_k N_j(a) N_k(a) X_j X_k x_i + \ldots\ldots,$$
$$(i = 1,\ 2,\ \ldots\ldots r).$$

The equations resulting from this summation will then be found to be identical with equations (1) above. Let $\bar{a}_1 = a_1 \neq 0$, $\bar{a}_2 = a_2$, $\bar{a}_3 = 2k\pi\sqrt{-1}$, where k is any integer $\neq 0$. Then $N_1(\bar{a})$ is infinite in all branches. Consequently $U_{\bar{a}}$ is not infinitesimal, and, therefore, $T_{\bar{a}}$ is not generated by an infinitesimal transformation of the group.

Let ϕ denote the matrix of the bilinear form $-\sum_1^r {}_l \sum_1^r {}_k \left(\sum_1^r {}_j a_j c_{jkl}\right) y_l z_k$, where the c_{jkl} are the structural constants defining any given structure, and let Δ denote the determinant of the matrix $\dfrac{e^{\phi}-1}{\phi}$ (in which case $\Delta = \prod_1^r {}_k \dfrac{e^{\rho_k}-1}{\rho_k}$, where $\rho_1 \ldots\ldots \rho_r$ are the roots of the characteristic equation of ϕ). Then, if A_{kj} denotes the first minor of Δ relative to the constituent in the j^{th} row and k^{th} column, the infinitesimal transformation which generates the parameter group belonging to the structure with which Δ is associated, is defined by the equations

$$a_k' = a_k + \sum_1^r {}_j \frac{A_{kj}}{\Delta} \alpha_j \delta t, \qquad (k = 1,\ 2,\ \ldots\ldots r), \tag{2}$$

where a_k and a_k' $(k = 1,\ 2,\ \ldots\ldots r)$ are the variables which enter into the equations of the parameter group, α_k $(k = 1,\ 2,\ \ldots\ldots r)$ are its parameters and δt is an infinitesimal.*

In a previous article I have partially stated a theorem in regard to Δ which will now be completed. I have shown, namely, that if the determinant Δ associated with any given structure does not vanish for any system of values of the a's, all groups of the corresponding structure are continuous, whereas, if Δ vanishes for certain systems of values of a's, some groups of the corresponding structure may be continuous and others discontinuous.† In order to complete the theorem it is necessary to prove that if the a's can be so chosen that $\Delta = 0$, at least one group of the corresponding structure will be discontinuous. This is proved as follows: Since the parameters $a_1 \ldots\ldots a_r$ are independent, the first

* Proc. Amer. Acad., Vol. 36, p. 102.

† Ibid., Vol. 36, p. 101.

minors of Δ cannot all contain Δ as a factor, that is to say, the first minors of Δ cannot all contain all of the factors which enter into the product $\Delta = \prod_{1}^{r}{}_k \frac{e^{\rho_k} - 1}{\rho_k}$. Consequently, if $\Delta = 0$, for certain values of the a's, one or more of the quotients $\frac{A_{kj}}{\Delta}$ must be infinite for these values of the a's. In this case the transformation (2) is no longer infinitesimal, and, therefore, the parameter group is discontinuous. If, then, $\bar{a}_1 \ldots \bar{a}_r$ is a system of values of the a's for which $\Delta = 0$, and we apply a finite transformation of the parameter group to the point whose coordinates are $\bar{a}_1 \ldots \bar{a}_r$, by properly choosing the parameters $a_1 \ldots a_r$, this point can be transformed into any other finite point whatever of the manifold $(a_1 \ldots a_r)$, but this finite transformation is not generated by an infinitesimal transformation of the group.

For example, consider the structure $(X_1, X_2) \equiv X_1$. The matrix ϕ in this case is

$$\phi \equiv \begin{vmatrix} a_2, & -a_1 \\ 0, & 0 \end{vmatrix}$$

and

$$\frac{e^{\phi} - 1}{\phi} \equiv \begin{vmatrix} \frac{e^{a_2} - 1}{a_2}, & -\frac{a_1}{a_2^2}(e^{a_2} - a_2 - 1) \\ 0, & 1 \end{vmatrix}.$$

Therefore,

$$\Delta \equiv \frac{e^{a_2} - 1}{a_2}.$$

[The roots of the characteristic equation of ϕ are $\rho_1 = a_2$, $\rho_2 = 0$, and, therefore, we also have $\Delta = \prod_{1}^{r}{}_k \frac{e^{\rho_k} - 1}{\rho_k} \equiv \frac{e^{a_2} - 1}{a_2}$]. Consequently, the equations defining the infinitesimal transformation of the parameter group belonging to the above structure are

$$\left.\begin{aligned} a_1' &= a_1 + \frac{a_2}{e^{a_2} - 1}\,\alpha_1 \delta t + \frac{a_1(e^{a_2} - a_2 - 1)}{a_2(e^{a_2} - 1)}\,\alpha_2 \delta t, \\ a_2' &= a_2 + \alpha_2 \delta t, \end{aligned}\right\} \qquad (3)$$

and the finite equations of this parameter group are found to be

$$\left.\begin{aligned} a_1' &= \frac{a_2 + \alpha_2}{e^{a_2 + \alpha_2} - 1}\left[e^{a_2}\frac{a_1}{a_2}(e^{a_2} - 1) + \frac{\alpha_1}{\alpha_2}(e^{\alpha_2} - 1)\right], \\ a_2' &= a_2 + \alpha_2. \end{aligned}\right\} \qquad (4)$$

If we apply this transformation to the point whose coordinates are $a_1 = \alpha_1$, $a_2 = 2k\pi\sqrt{-1}$, where k is any integer $\neq 0$, by a proper choice of the parameters α_1, α_2, this point can be transformed into any finite point whatever of the manifold (a_1, a_2). But for these values of the a's, $\Delta = 0$; consequently, the transformation (3) is not infinitesimal, and, therefore, the finite transformation (4) is not generated by an infinitesimal transformation of the group.

To each r-parameter complex group, G_r, there corresponds a definite r-parameter real group g_r, the properties of which are closely related to those of G_r.* It is possible, however, for g_r to be continuous and G_r discontinuous.† But if a group is continuous, two points of general position on the same smallest invariant manifold relative to that group can always be *continuously* interchanged by the transformations of the group, whereas Rettger has shown that if a group is discontinuous, two such points cannot always be so interchanged; that is to say, cannot always be interchanged by a transformation of the group which can be generated by an infinitesimal transformation of the group.‡ For all the discontinuous complex groups, G_r, cited by Rettger to illustrate this statement, the corresponding real groups, g_r, are continuous. Thus it appears that the exception imposed upon Lie's chief theorem by the possibility of discontinuity in a group with continuous parameters necessitates a restriction upon the relation between the transitivity of a complex group and that of its corresponding real group.

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* Lie, Transformationsgruppen, Vol. 3, p. 362.

† E. g. consider the three-parameter group defined by the equations

$$x_1' = x_1 e^{a_1} + a_1^2 e^{a_1} x_2 + 2a_1 e^{a_1} x_3 + a_2,$$
$$x_2' = x_2 e^{a_1}.$$
$$x_3' = a_1 e^{a_1} x_2 + e^{a_1} x_3 + a_3.$$

The infinitesimal transformation which generates the finite transformation T_a defined by these equations, is found to be

$$U_a \equiv a_1 \left(2x_3 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_3} + x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} \right) + \frac{a_1}{e^{a_1} - 1} \left(a_2 + 2a_3 - \frac{2a_1 a_3 e^{a_1}}{e^{a_1} - 1} \right) \frac{\partial}{\partial x_1} + \frac{a_1 a_3}{e^{a_1} - 1} \frac{\partial}{\partial x_3}.$$

For all real values of the a's, U_a is infinitesimal, and, consequently, the real group g_r, of transformations T_a is continuous. But for the complex values $a_1 = 2k\pi\sqrt{-1}$, $a_2 = \alpha_2$, $a_3 = \alpha_3 \neq 0$, where k is any integer $\neq 0$, U_a is no longer infinitesimal, and, consequently, the complex group, G_r, of transformations T_a is discontinuous.

‡ Amer. Jour., Vol. XXII, pp. 90–94.

Several structures which are of the same type for complex groups may constitute entirely distinct types of structure for real groups.* This follows from the fact that for real groups the structural constants defining any given structure must all be real, and two real structures can only be said to belong to the same type, when one can be transformed into the other by a real transformation.†

Suppose, then, that we have given two structures which are of the same type, A, for complex groups, but which constitute distinct types, B and C, for real groups. If the determinant Δ, associated with type A, does not vanish for any system of values of the parameters, *all* groups of this type are continuous, and, consequently, all real groups of types B and C are continuous as well. If, however, the determinant Δ, associated with type A, vanishes for certain systems of values of the parameters, one of the following cases may occur with respect to the continuity of the real groups of types B and C:

1. All real groups of both types B and C are continuous.
2. All real groups of both types B and C are discontinuous.
3. All real groups of one type, B, are continuous, and one or more, but not all, real groups of the other type, C, are discontinuous.
4. All real groups of one type, B, are continuous, and all real groups of the other type, C, are discontinuous.

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* E. g. consider the four structures

$$1.\quad \begin{cases} (X_1, X_2) \equiv X_3, & (X_1, X_3) \equiv -X_2 \quad, \quad (X_2, X_3) \equiv X_1, \\ (X_1, X_4) \equiv 0\ , & (X_2, X_4) \equiv 0 \quad, \quad (X_3, X_4) \equiv 0\ . \end{cases}$$

$$2.\quad \begin{cases} (X_1, X_2) \equiv X_1, & (X_1, X_3) \equiv 2X_2 \quad, \quad (X_2, X_3) \equiv X_3, \\ (X_1, X_4) \equiv 0\ , & (X, X_4) \equiv 0 \quad, \quad (X_3, X_4) \equiv 0\ . \end{cases}$$

$$3.\quad \begin{cases} (X_1, X_2) \equiv 0\ , & (X_1, X_3) \equiv 0 \quad, \quad (X_2, X_3) \equiv X_4, \\ (X_1, X_4) \equiv 0\ , & (X_2, X_4) \equiv X_1 + X_3, \quad (X_3, X_4) \equiv X_2. \end{cases}$$

$$4.\quad \begin{cases} (X_1, X_2) \equiv -X_4 & (X_1, X_3) \equiv -X_4, \quad (X_2, X_3) \equiv 0\ , \\ (X_1, X_4) \equiv X_3, & (X_2, X_4) \equiv X_1 \quad, \quad (X_3, X_4) \equiv X_1. \end{cases}$$

It will be found that each of the above structures can be transformed into any one of the others, but only by means of a complex transformation. Thus the transformation

$$X_1 \equiv -X_1' - X_3' \sqrt{-1}, \quad X_2 \equiv X_2' \sqrt{-1}, \quad X_3 \equiv X_3' \sqrt{-1} - X_1', \quad X_4 \equiv X_4'$$

transforms (1) into (2), the transformation

$$X_1 \equiv X_4', \quad X_2 \equiv X_2', \quad X_3 \equiv \frac{\sqrt{-1}}{2}(X_1' - X_3') - X_4', \quad X_4 \equiv -\frac{\sqrt{-1}}{2}(X_1' + X_3')$$

transforms (2) into (3), etc. Consequently, the above four structures are of the same type for complex groups but constitute distinct types of structure for real groups.

† Transformations gruppen, Vol. 3, p. 361; Proc. Amer. Acad., Vol. 36, p. 105.

The following four examples show the possibility of the occurrence of each of the above four cases:

Case 1. Consider the structures

$$\begin{cases}(X_1, X_2) \equiv 0 \quad , & (X_1, X_3) \equiv 0 \quad , \quad (X_2, X_3) \equiv X_1, \\ (X_1, X_4) \equiv 2X_1, & (X_2, X_4) \equiv X_2, \quad (X_3, X_4) \equiv 2X_2 + X_3,\end{cases}$$

and

$$\begin{cases}(X_1, X_2) \equiv 0 \quad , & (X_1, X_3) \equiv X_1 + 2X_4, \quad (X_2, X_3) \equiv 2X_2, \\ (X_1, X_4) \equiv X_2, & (X_2, X_4) \equiv 0 \quad\quad\quad , \quad (X_3, X_4) \equiv - X_4.\end{cases}$$

These are of the same type, A, for complex groups but constitute distinct types, B and C, for real groups, since they can only be interchanged by means of a complex transformation. The adjoined complex group of type A is discontinuous, and, consequently, all complex groups of that type are discontinuous.* The determinants Δ_1 and Δ_2, associated with these two structures, are respectively

$$\Delta_1 \equiv \frac{(e^{a_4} - 1)^2(e^{2a_4} - 1)}{2a_4^3}, \quad \Delta_2 \equiv \frac{(e^{a_3} - 1)^2(e^{2a_3} - 1)}{2a_3^3}.$$

Since neither of these determinants vanishes for any system of real values of the a's, all real groups of both types B and C are continuous.

Case 2. Consider the structures

$$(X_1, X_2) \equiv X_3 \quad , \quad (X_1, X_3) \equiv - X_2, \quad (X_2, X_3) \equiv X_1 \quad ,$$

and

$$(X_1, X_2) \equiv - 2X_1, \quad (X_1, X_3) \equiv X_2 \quad , \quad (X_2, X_3) \equiv - 2X_3,$$

which are of the same type for complex groups but constitute distinct types for real groups. The adjoined complex group of type A is discontinuous, and, consequently, all complex groups of this type are discontinuous. The determinants Δ_1 and Δ_2, associated with these two structures, are respectively

$$\Delta_1 \equiv \frac{(e^{\sqrt{-(a_1^2+a_2^2+a_3^2)}} - 1)(e^{-\sqrt{-(a_1^2+a_2^2+a_3^2)}} - 1)}{a_1^2 + a_2^2 + a_3^2},$$

$$\Delta_2 \equiv \frac{(e^{2\sqrt{a_2^2+a_1a_3}} - 1)(e^{-2\sqrt{a_2^2+a_1a_3}} - 1)}{-4(a_2^2 + a_1a_3)}.$$

Δ_1 vanishes if $a_1^2 + a_2^2 + a_3^2 = 4k^2\pi^2$; Δ_2 vanishes if $a_2^2 + a_1a_3 = -k^2\pi^2$, where k is any integer $\neq 0$, and each of these relations can evidently be satisfied by real values of a_1, a_2, a_3. Moreover, the adjoined real groups of both types B and C are discontinuous, and, consequently, all real groups of both of these types are discontinuous.

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* Taber, Bull. Amer. Math. Soc., Feb., 1900, p. 203.

Case 3. The structures

$$(X_1, X_2) \equiv 0, \quad (X_1, X_3) \equiv X_2, \quad (X_2, X_3) \equiv -X_1,$$

and

$$(X_1, X_2) \equiv 0, \quad (X_1, X_3) \equiv X_1, \quad (X_2, X_3) \equiv X_2$$

can also be shown to be of the same type, A, for complex groups, but to constitute distinct types, B and C, for real groups.

The determinant Δ, associated with type A, vanishes for certain systems of complex values of the a's, but the adjoined complex group of this type is continuous. Consequently, one or more, but not all, complex groups of type A are discontinuous. The determinants Δ_1 and Δ_2, associated with types B and C, are respectively

$$\Delta_1 \equiv \frac{(e^{a_3\sqrt{-1}} - 1)(e^{-a_3\sqrt{-1}} - 1)}{a_3^2}, \quad \Delta_2 \equiv \left(\frac{e^{a_3} - 1}{a_3}\right)^2.$$

Δ_1 vanishes for the real values $a_3 = 2k\pi$, where k is any integer $\neq 0$, but the adjoined real group of this type is continuous. Consequently, one or more, but not all, real groups of this type are discontinuous. Δ_2 does not vanish for any system of real values of the parameters, and, consequently, all real groups of the second type of structure are continuous.

Case 4. Consider structures 3 and 4 in the foot-note to page 12, which were there shown to be of the same type, A, for complex groups, but to constitute distinct types, B and C, for real groups. The adjoined complex group of type A is discontinuous; consequently, all complex groups of this type are discontinuous. The determinants Δ_3 and Δ_4, associated with the structures under consideration, are respectively

$$\Delta_3 \equiv \frac{(e^{a_3\sqrt{-1}} - 1)(e^{-a_3\sqrt{-1}} - 1)}{a_3^2}, \quad \Delta_4 \equiv \frac{(e^{a_4} - 1)(e^{-a_4} - 1)}{-a_4^2}.$$

Δ_4 does not vanish for any system of real values of the a's, and, therefore, all real groups of type 4 are continuous. Δ_3, however, vanishes for the real values $a_3 = 2k\pi$, where k is any integer $\neq 0$. Moreover, the real adjoined group of type 3 is discontinuous, and, therefore, all real groups of this type are discontinuous.*

* The symbols of infinitesimal transformation of this real adjoined group are

$$-x_3\frac{\partial}{\partial x_4} - x_4\frac{\partial}{\partial x_1} - x_4\frac{\partial}{\partial x_3}, \quad x_2\frac{\partial}{\partial x_4} - x_4\frac{\partial}{\partial x_2}, \quad x_2\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_3} + x_3\frac{\partial}{\partial x_2}.$$

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